## NUMBER THEORY AND DIOPHANTINE ANALYSIS.

**145. Proposed by J. D. WILLIAMS, being the 12th of his fourteen challenge problems proposed in 1832.**

Make $x^2+y^2=\square$, $\frac{5}{4}(x^2+y^2)=$a cube, $xy=2x^3$, $2(x+y)+\dfrac{xy}{x+y}=\square$, and $(x^4+y^4)(x^2+y^2)-(x^5+y^5)\sqrt{(x^2+y^2)}=\square$.

No solution has been received.

**146. Proposed by PROFESSOR JOSE J. CORONADO, Halapa, Vericruz, Mexico.**

Find two numbers whose difference is equal to the difference of their cubes.

**I. Solution by G. B. M. ZERR, Philadelphia, Pa.; A. H. HOLMES, Brunswick, Me.; and J. E. SANDERS, Reinersville, O.**

Let $x$, $y$ be the numbers. Then $x-y=x^3-y^3$, $1=x^2+xy+y^2$, if $x\neq y$. Let $x=vy$.

$$\therefore y=\frac{1}{\sqrt{(v^2+v+1)}},\ x=\frac{v}{\sqrt{(v^2+v+1)}}.$$

Let $v^2+v+1=(nv+1)^2$.

$$\therefore v=\frac{1-2n}{n^2-1}.\quad \therefore x=\frac{1-2n}{n^2-n+1},\ \text{and } y=\frac{n^2-1}{n^2-n+1},$$

where $n$ can have any value, positive or negative, whole or fractional.

**II. Solution by DR. L. E. DICKSON, The University of Chicago.**

$x-y=x^3-y^3$. Say $x\neq y$. $\therefore 1=x^2+xy+y^2$.

If any two numbers are desired, there are an infinitude of answers. If two integers are desired (neither zero), then one must be negative, otherwise $x^2+xy+y^2 \geqq 3$. Say $y$ is negative, $=-z$. $\therefore 1=x^2-xz+z^2$, $x$ and $z$ positive integers; $\therefore 1-xz=(x-z)^2$; $\therefore 1-xz=0$, or positive.

$xz \leqq 1$. $\therefore x=z=1$. If one is zero, the other$=0$ or $\pm1$.

$\therefore$ Only integral sets are $(0, 0)$, $(0, \pm1)$, $(\pm1, 0)$, $(\pm1, \mp1)$.

Combined: $(x, y)$, $\begin{matrix} x=0, & \pm1, \\ y=0, & \pm1. \end{matrix}$

## MISCELLANEOUS.

**170. Proposed by J. W. NICHOLSON, A. M., LL. D., Baton Rouge, La.**

If $n$ and $m$ are any two real numbers whatever, $n$ being less than $m$, find a rational $r$ such that $\sqrt{n}<r<\sqrt{m}$.

**Solution by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.**

This depends on how we choose our markings. If we choose the natural numbers 1, 2, 3, 4, etc., $\sqrt{n}$ may be defined by two infinite series of rational numbers, and $\sqrt{m}$ may also be so defined. As these two infinite

series cannot be the same when $n \neq m$, there is some rational number between them.

Let $\sqrt{n}=e+f$, where $f$ is the decimal part, and $\sqrt{m}=g+h$, where $h$ is the decimal part. Then $r=e+k$, where $k$ is any number from 1 to $g-e-1$.

**171. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.**

If $\lim\limits_{x=a} \dfrac{\phi(x)}{\psi(x)}=\lambda$, show $\lim\limits_{x=a} \left[\dfrac{\lambda}{\phi(x)}-\dfrac{1}{\psi(x)}\right]=\dfrac{\lambda\psi''(a)-\phi''(a)}{2\phi'(a)\psi'(a)}$.

**Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.**

By hypothesis, $\phi(a)=0$, and $\psi(a)=0$. We must *assume* that $\phi'(a)\neq 0$ and $\psi'(a)\neq 0$. Then

$$\text{Lim.}\frac{\phi(x)}{\psi(x)}=\left[\frac{\phi'(x)}{\psi'(x)}\right]_{x=a}=\lambda. \quad \therefore \lambda\psi'(a)-\phi'(a)=0.$$

$$\text{Lim.}\left[\frac{\lambda}{\phi(x)}-\frac{1}{\psi(x)}\right]=\text{Lim.}\left[\frac{\lambda\psi(x)-\phi(x)}{\phi(x)\psi(x)}\right]=\text{Lim.}\left[\frac{\lambda\psi'(x)-\phi'(x)}{\phi(x)\psi'(x)+\phi'(x)\psi(x)}\right]$$

$$=\text{Lim.}\left[\frac{\lambda\psi''(x)-\phi''(x)}{\phi(x)\psi''(x)+2\phi'(x)\psi'(x)+\phi''(x)\psi(x)}\right]=\frac{\lambda\psi''(a)-\phi''(a)}{2\phi'(a)\psi'(a)}.$$

Also solved similarly by G. B. M. Zerr. Unless one assumes that $\phi(a)=\psi(a)=0$, the problem is not true, as may be easily verified. ED. F.

## PROBLEMS FOR SOLUTION.

### ALGEBRA.

**297. Proposed by W. J. GREENSTREET, Marling School, Stroud, England.**

If $a$, $b$, $c$, $d$, $f$, $g$, $h$ are all real, and $a$, $ab-h^2$, $abc+2fgh-af^2-bg^2-ch^2$ are all positive, show that $b$, $c$, $bc-f^2$, and $ca-g^2$ are also positive.

### GEOMETRY.

**330. Proposed by J. J. QUINN, Ph. D., New Castle, Pa.**

A line pivoted at the origin revolving with a constant angular velocity, intersects another moving parallel to the $Y$-axis with a constant linear velocity. (1) Find the locus of their intersection when the ratio of their velocities is as $m:n$ referred to a quadrant and a radius, respectively. (2) Assume $m=3$ and $n=2$, and apply to the trisection of an angle. (3) Under what conditions will this curve become a quadratrix? (4) Name the curve.